The Spectrum of Steady State Turbulent Convection *

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Based on Heisenberg's statistical theory of turbulence, a model for steady state turbulent convection is herein proposed, and on the basis of this model, equations for the energy spectrum for steady state turbulent convection are derived. The spectrum is obtained from the solution of a nonlinear integral equation. After the integral equation is brought into a universally valid nondimensional form, it is transformed into a nonlinear first order differential equation to be solved numerically, with the Rayleigh number appearing as the only parameter. The energy spectrum has a substantial deviation from the Kolmogoroff law, as a result of the buoyancy force acting on the rising and falling eddies. The presented theory may be applicable to convection in planetary and stellar atmospheres wherein the radiative heat transport is small.

Introduction

Without justification, it is a widely believed view that the spectrum of atmospheric turbulence can be satisfactorily described by the universal Kolmogoroff-Heisenberg law [1,2] for isotropic steady state turbulence. This law predicts a $k^{-5/3}$ power dependence for the equilibrium range and, where viscosity becomes important, for large wave numbers it is followed by a k^{-7} tail. Apart from the fact that this law is valid only for the case of steady state turbulence, which in general is not given in the earth atmosphere, it cannot be applied to describe the spectrum of turbulent convection because it does not take into account the buoyancy force which is relevant for the phenomena of convection. Since the buoyancy force acts primarily on large eddies, which contribute mostly to the small wave number range of the energy spectrum, one would expect that the deviation between the Kolmogoroff-Heisenberg law and the observed spectrum of atmospheric turbulence increases with decreasing wave numbers.

The phenomena of turbulent convection is not only of great importance in astrophysics in the theory of stellar structure but also for planetary atmospheres, in particular the atmosphere of the earth.

It should be mentioned that the phenomena of steady state turbulent convection can be indirectly observed by atmospheric scintillation above a hot surface and also directly in the process of boiling.

Because of the great significance of the problem of turbulent convection in understanding stellar atmospheres (in particular the solar atmosphere), a theory for the energy spectrum of turbulent convection which is based on Heisenberg's statistical theory of turbulence has been developed by LEDOUX, SCHWARZSCHILD and SPIEGEL [3]. This theory neglects turbulent heat transport and is therefore only applicable to the limiting case of vanishing Prandtl number, which is a good approximation for the solar atmosphere because of the predominance of radiative energy transfer over turbulent heat transport. As expected, the calculated spectrum approaches the Kolmogoroff-Heisenberg law asymptotically for large wave numbers and has a substantial deviation from the $k^{-5/3}$ power law for small wave numbers, being replaced by a k^{-7} dependence. Accordingly the spectrum is peaked near the lower wave number cutoff of the energy spectrum and the energy is primarily concentrated in the largest eddies. The theory is in qualitative agreement with the observation, especially confirming the peaking of the spectrum for small wave numbers.

Since the temperatures of planetary atmospheres are comparatively low, the Prandtl number cannot be considered to be small. The same will also be true for very cool stellar atmospheres where the theory by Ledoux et al. breaks down.

We will present in this paper a theory of turbulent convection which is better adjusted to the physical situation existing in a relatively cool atmo-

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[1] W. HEISENBERG, Z. Physik **124**, 628 [1948] and Proc. Roy. Soc. London A **195**, 402 [1948].

[2] S. CHANDRASEKHAR, Proc. Roy. Soc. London A **200**, 20 [1949].

[3] P. LEDOUX, M. SCHWARZSCHILD, and E. A. SPIEGEL, Astrophys. J. **133**, 184 [1961].

sphere, such as in the terrestrial atmosphere. As in the theory by Ledoux et al., we will also base our assumptions on Heisenberg's statistical theory of turbulence. However, to Ledoux et al.'s we must introduce an additional assumption relating to the heat transport by turbulent eddies with the energy spectrum.

1. The Basic Equations

We will consider only convection phenomena with small Mach numbers, excluding the possible occurrence of shock waves. The assumption of small Mach numbers is generally a very good one for planetary atmospheres, for the solar and stellar atmospheres, however, this is not too certain.

Under this assumption we can make use of the Boussinesq approximation [4] with the following equations of motion:

$$\partial \boldsymbol{v}/\partial t + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -(1/\varrho)\,\nabla p' + \nu\,\nabla^2\boldsymbol{v} + g\,\alpha\,T'\,\boldsymbol{e}_z\,, \tag{1.1}$$

the energy equation

$$\partial T'/\partial t + \boldsymbol{v}\cdot\nabla T' = \chi\,\nabla^2 T' + \beta\,\boldsymbol{v}\cdot\boldsymbol{e}_z\,, \tag{1.2}$$

and the continuity equation

$$\operatorname{div}\boldsymbol{v} = 0\,. \tag{1.3}$$

In Eqs. (1.1) – (1.3) T' and p' are the perturbations of the temperature and pressure fields, from their respective equilibrium values, α is the thermal expansion coefficient, $\beta = -\Delta\nabla T$ is the excess of the temperature gradient over the adiabatic temperature gradient, $\boldsymbol{e}_z$ is a unit vector in the vertical direction, and χ is the heat conduction coefficient.

To derive the equations for the energy spectrum of turbulent convection, we proceed as follows:

a) We expand the fields of velocity, pressure and temperature in wave number space

$$\boldsymbol{v}(\boldsymbol{r},t) = \sum_k \boldsymbol{v}(\boldsymbol{k},t)\,\exp\{i\,\boldsymbol{k}\cdot\boldsymbol{r}\}\,, \tag{1.4}$$

$$p'(\boldsymbol{r},t) = \sum_k p'(\boldsymbol{k},t)\,\exp\{i\,\boldsymbol{k}\cdot\boldsymbol{r}\}\,, \tag{1.5}$$

$$T'(\boldsymbol{r},t) = \sum_k T'(\boldsymbol{k},t)\,\exp\{i\,\boldsymbol{k}\cdot\boldsymbol{r}\}\,. \tag{1.6}$$

b) We will assume isotropy of the turbulent motion which implies that we can, without loss of information, average all quantities over a spherical surface in $\boldsymbol{k}$-space.

c) We then consider the following spectral functions:

$$F(\boldsymbol{k},t) = \frac{V}{8\,\pi^3}\,k^2 \int \langle v(\boldsymbol{k},t)\,v^*(\boldsymbol{k},t)\rangle\,\mathrm{d}\Omega\,, \tag{1.7}$$

$$G(\boldsymbol{k},t) = \frac{V}{8\,\pi^3}\,k^2 \int \langle T'(\boldsymbol{k},t)\,T^{*\prime}(\boldsymbol{k},t)\rangle\,\mathrm{d}\Omega\,, \tag{1.8}$$

$$H(\boldsymbol{k},t) = \frac{1}{2}\,\frac{V}{8\,\pi^3}\,k^2\,\boldsymbol{e}_z\cdot\int\{\langle\boldsymbol{v}(\boldsymbol{k},t)\,T^{*\prime}(\boldsymbol{k},t)\rangle + \langle v^*(\boldsymbol{k},t)\,T'(\boldsymbol{k},t)\rangle\}\,\mathrm{d}\Omega. \tag{1.9}$$

In Eqs. (1.7) – (1.9) the quantities in $\langle\ \rangle$ brackets denote ensemble averages.

d) We transform Eqs. (1.1) and (1.2) into $\boldsymbol{k}$-space, multiplying the resulting equations with $\boldsymbol{v}^*(\boldsymbol{k},t)$ resp. $T^{*\prime}(\boldsymbol{k},t)$, $p^{*\prime}(\boldsymbol{k},t)$, and after taking the ensemble averages integrate over a spherical surface in $\boldsymbol{k}$-space. With the definitions (1.7) – (1.9) we then obtain the following set of equations for the spectral functions $F(\boldsymbol{k},t)$, $G(\boldsymbol{k},t)$ and $H(\boldsymbol{k},t)$:

$$-\partial F/\partial t = 2\,\nu\,k^2\,F - 2\,g\,\alpha\,H - \int_{k_0}^{\infty} Q(k,k')\,\mathrm{d}k', \tag{1.10}$$

$$-\partial G/\partial t = 2\,\chi\,k^2\,G - 2\,\beta\,H - \int_{k_0}^{\infty} U(k,k')\,\mathrm{d}k'. \tag{1.11}$$

In (1.10) and (1.11) k_0 is a lower cutoff wave number below which it is assumed that the spectral functions are zero. k_0 is related to the largest possible wave number compatible with the boundary conditions.

The terms involving $Q(k,k')$ and $U(k,k')$ result from the nonlinear interaction terms $(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}$ and $\boldsymbol{v}\cdot\nabla T'$, and are trilinear in the Fourier-transform of the velocity field resp. bilinear in the temperature and linear in the velocity fluctuation.

In deriving (1.10) we observed the vanishing of the velocity pressure correlation for the case of isotropic turbulence.

e) In order to see the connection to Heisenberg's theory we put $g=0$. The spectrum $F(k,t)$ which is the energy spectrum is then determined from Eq. (1.10) above. For $g=0$ Eq. (1.10) is given by

$$-\partial F/\partial t = 2\,\nu\,k^2\,F - \int_{k_0}^{\infty} Q(k,k')\,\mathrm{d}k'. \tag{1.12}$$

[4] E. A. Spiegel and G. Veronis, Astrophys. J. **131**, 442 [1960].

We integrate Eq. (1.12) over k from $k=k_0$ to k with the result

$$-\frac{\partial}{\partial t}\int_{k_0}^{k} F(k',t)\,\mathrm{d}k' = 2\,\nu\int_{k_0}^{k} F(k',t)\,k'^2\,\mathrm{d}k' - \int_{k_0}^{k}\mathrm{d}k'\int_{k_0}^{\infty} Q(k',k'')\,\mathrm{d}k''. \tag{1.13}$$

If we compare Eq. (1.13) with Heisenberg's equation for $F(k,t)$

$$-\frac{\partial}{\partial t}\int_{k_0}^{k} F(k',t)\,\mathrm{d}k = 2\left(\nu+\varkappa\int_{k}^{\infty}\sqrt{\frac{F(k'',t)}{k''^3}}\,\mathrm{d}k''\right)\int_{k_0}^{k} F(k',t)\,k'^2\,\mathrm{d}k', \tag{1.14}$$

we see that Heisenberg's theory is based on the ad hoc assumption

$$\int_{k_0}^{k}\mathrm{d}k'\int_{k_0}^{\infty} Q(k',k'')\,\mathrm{d}k'' = -2\,\nu(k)\int_{k_0}^{k} F(k',t)\,k'^2\,\mathrm{d}k', \tag{1.15}$$

where $\nu(k)$ is interpreted as an "eddy viscosity" for which Heisenberg makes the assumption

$$\nu(k) = \varkappa\int_{k}^{\infty}\sqrt{\frac{F(k')}{k'^3}}\,\mathrm{d}k', \tag{1.16}$$

$\varkappa$ is a universal constant and measurements have lead to a numerical value of $\varkappa\approx 1/3$.

The significance of Heisenberg's assumption (1.15) is derived from the fact that it makes it relatively easy to obtain the energy spectrum as a solution of Eq. (1.14) which would otherwise be extremely difficult. Since no better hypothesis seems to be at hand, we will postulate Heisenberg's assumption for Eq. (1.10).

f) In order to obtain a similar reasonable assumption for the nonlinear integral term in Eq. (1.11), we postulate in analogy to Eq. (1.15).

$$\int_{k_0}^{k}\mathrm{d}k'\int_{k_0}^{\infty} U(k',k'')\,\mathrm{d}k'' = -2\,\chi(k)\int_{k_0}^{k} G(k')\,k'^2\,\mathrm{d}k', \tag{1.17}$$

where $\chi(k)$ is an "eddy heat conduction coefficient" analogous to the "eddy viscosity" defined by Heisenberg.

In analogy to Heisenberg's assumption (1.16), it is required to relate $\chi(k)$ in some physically meaningful way to the spectral functions. How this is to be done most expediently is discussed below.

g) In order that the system of Eqs. (1.10) – (1.11) has a solution, it must be supplemented by an expression relating two of the occurring spectral functions. In the theory of Ledoux et al. it was assumed that the velocity and temperature fluctuations are in phase and it was made plausible that under this assumption one should have

$$H = \sqrt{\tfrac{1}{2}FG}. \tag{1.18}$$

It should be pointed out that this assumption was made with no reference to the special conditions prevailing in the solar atmosphere; it seems therefore reasonable to assume that the same relation is valid in our situation.

h) For the following it is more convenient to introduce instead of $\nu(k)$ and $\chi(k)$ different turbulent transport coefficients defined by

$$\nu_\varepsilon(k) = \frac{1}{k^2 F(k)}\frac{\mathrm{d}}{\mathrm{d}k}\left[\nu(k)\int_{k_0}^{k} F(k')\,k'^2\,\mathrm{d}k'\right], \tag{1.19}$$

$$\chi_\varepsilon(k) = \frac{1}{k^2 G(k)}\frac{\mathrm{d}}{\mathrm{d}k}\left[\chi(k)\int_{k_0}^{k} G(k')\,k'^2\,\mathrm{d}k'\right]. \tag{1.20}$$

With these definitions and the assumptions (1.15) and (1.17) one has finally for the equations of the spectral functions

$$-\partial F/\partial t = 2\,\{\nu+\nu_\varepsilon(k)\}\,k^2 F - 2\,g\,\alpha\,H, \tag{1.21}$$

$$-\partial G/\partial t = 2\,\{\chi+\chi_\varepsilon(k)\}\,k^2 G - 2\,\beta\,H, \tag{1.22}$$

$$H = \sqrt{\tfrac{1}{2}FG}. \tag{1.23}$$

i) We return now to the question of how to relate $\chi(k)$ to the spectral functions. For this we use the analogy of mixing eddies and mixing gas molecules. From kinetic theory of mixing gas molecules a relation between χ and ν is derived given by

$$\chi = \nu/\gamma, \tag{1.24}$$

where γ is the specific heat ratio for which, depending on the number of degrees of freedom, one has $4/3<\gamma<5/3$.

We may assume that relation (1.24) remains also valid in relating $\nu(k)$ and $\chi(k)$; that is, by assuming

$$\chi(k) = \nu(k)/\gamma^*, \tag{1.25}$$

where γ^* is the "specific heat ratio for eddies", which may be assumed of the order unity. It turns out to be more expedient, however, to postulate such a relation for the quantities $\nu_\varepsilon(k)$ and $\chi_\varepsilon(k)$ defined by (1.19) and (1.20), because only in this case the integral equation for the energy spectrum $F(k)$ of steady state turbulent convection takes a particular simple form most tractable by analytical and numerical methods. A further simplification results by assuming $\gamma \cong \gamma^*$ which should be not too bad an assumption for this rather crude theory. We therefore postulate

$$\chi_\varepsilon(k) = \nu_\varepsilon(k)/\gamma^* = \nu_\varepsilon(k)/\gamma . \qquad (1.26)$$

It should be pointed out that from a physical point of view the difference resulting from either assuming (1.25) or (1.26) should be not too great since the observed energy spectra follow mostly power laws and it can be seen that in this case $\nu_\varepsilon(k) = \nu(k)$ and $\chi_\varepsilon(k) = \chi(k)$ is fulfilled exactly.

j) For steady state turbulent convection we have to put $\partial/\partial t = 0$ and obtain from (1.21) – (1.23)

$$\nu^* k^2 F = g\,\alpha\,H , \qquad (1.27)$$

$$\chi^* k^2 G = \beta\,H , \qquad (1.28)$$

$$H = \sqrt{\tfrac{1}{2} F G} , \qquad (1.29)$$

with the definitions

$$\nu^* = \nu + \nu_\varepsilon(k) , \qquad (1.30)$$

$$\chi^* = \chi + \chi_\varepsilon(k) . \qquad (1.31)$$

Multiplying both sides of (1.27) and (1.28) and eliminating H with the aid of the Eq. (1.29) results in

$$2\,\nu^* \chi^* k^4 = g\,\alpha\,\beta . \qquad (1.32)$$

Because of (1.24) and (1.26) we have $\chi^* = \nu^*/\gamma$ thus that one can write for Eq. (1.32)

$$\nu^* k^2 = (g\,\alpha\,\beta\,\gamma/2)^{1/2} . \qquad (1.33)$$

For $g = 0$ one has

$$\nu^* = 0 , \qquad (1.34)$$

respectively

$$\nu + \frac{1}{k^2 F}\frac{\mathrm{d}}{\mathrm{d}k}\left[\nu(k)\int_{k_0}^{k} F\,k^2\,\mathrm{d}k\right] = 0 \qquad (1.35)$$

from which Heisenberg's equation for steady state turbulence in the absence of gravitational buoyancy is obtained by multiplying with $F(k)$ and integrating over k from $k = k_0$ to k. It seems, therefore, that Eq. (1.33) is the most simple generalization of Heisenberg's theory in the presence of a constant gravitational force.

k) We have finally to say something about the lower wave number cutoff. Ledoux et al. make it plausible that the turbulent convection is peaked around the cone in $\boldsymbol{k}$-space

$$k_x^2 + k_y^2 = k_z^2 = \tfrac{1}{2} k^2 . \qquad (1.36)$$

Without giving proof, we would like to mention that this condition can also be derived more systematically [5]. In a distance d above the ground level, the smallest wave number k_z must obey the boundary condition

$$k_z = \pi/d . \qquad (1.37)$$

By substituting (1.37) into (1.36), one obtains from this a lower cutoff wave number k_0 given by

$$k_0 = \sqrt{2}\,\pi/d . \qquad (1.38)$$

2. Nondimensional Form of the Integral Equation for the Energy Spectrum and its Transformation into a Nonlinear Differential Equation

We introduce a nondimensional wave number q defined by

$$k = k_0\,q , \qquad (2.1)$$

and a nondimensional energy spectrum $f(q)$ defined by

$$F(k) = g\,\alpha\,\beta\,\gamma \cdot f(q)/2\,\varkappa^2 k_0^3 . \qquad (2.2)$$

With these substitutions Eq. (1.33) reads

$$\delta - \frac{1}{q^2} + \frac{1}{f\,q^2}\frac{\mathrm{d}}{\mathrm{d}q}\left[\int_q^\infty \sqrt{\frac{f}{q^3}}\,\mathrm{d}q \int_1^q f\,q^2\,\mathrm{d}q\right] = 0 , \qquad (2.3)$$

where δ is given by

$$\delta = \nu\,k_0^2/(g\,\alpha\,\beta\,\gamma/2)^{1/2} . \qquad (2.4)$$

δ can be also expressed in terms of the Rayleigh number

$$R = g\,\alpha\,\beta\,d^4/\chi\,\nu = g\,\alpha\,\beta\,\gamma\,d^4/\nu^2 = 4\,\pi^4 g\,\alpha\,\beta\,\gamma/\nu^2 k_0^4 , \qquad (2.5)$$

hence,

$$\delta = 2\,\pi^2\sqrt{2/R} . \qquad (2.6)$$

Of special importance is the case $R \to \infty$ for which $\delta = 0$. After differentiating the third term occurring in (2.3), one has

$$\delta - \frac{1}{q^2} + \int_q^\infty \sqrt{\frac{f}{q^3}}\,\mathrm{d}q - \frac{1}{\sqrt{f\,q^7}}\int_1^q f\,q^2\,\mathrm{d}q = 0 . \qquad (2.7)$$

[5] F. Winterberg, Heisenberg's Statistical Theory of Turbulence and the Equations of Motion for a Turbulent Flow, Z. Naturforsch. **23 a**, 1471 [1968].

A solution of Eq. (2.7) is suggested by CHANDRASEKHAR's [2] solution of Heisenberg's equation. In accordance with this solution we make the substitution

$$y = f\, q^3\,, \tag{2.8}$$

$$x = \int_1^q f\, q^2\, \mathrm{d}q\,. \tag{2.9}$$

Differentiation of the transformed equation with respect to x yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4\sqrt{y}}{x} - \frac{4\,y}{x} + 4 = 0\,. \tag{2.10}$$

In deriving Eq. (2.10) we used the relation

$$\frac{\mathrm{d}x}{\mathrm{d}q} = f\, q^2\,, \tag{2.11}$$

obtained by differentiating (2.9).

From Eq. (2.8) and (2.11) we obtain

$$\mathrm{d}x/\mathrm{d}q = y/q\,. \tag{2.12}$$

From (2.9) follows furthermore that for $q=1$, $x=0$.

The differential Eq. (2.10) has to be solved numerically, resulting in a function $y = y(x, A)$, where A is a constant of integration. Inserting this function into (2.12), one obtains by integrating with the boundary condition $q=1$ for $x=0$:

$$q = \exp\left[\int_0^x \mathrm{d}x/y(x, A)\right]. \tag{2.13}$$

From (2.13) follows $q = q(x, A)$ which, together with the solution of the differential Eq. (2.10) $y = y(x, A)$, determines $y = y(q, A)$ and hence according to (2.8)

$$f(q) = y(q, A)/q^3. \tag{2.14}$$

What remains to be done is to relate the constant of integration occurring in the solution of (2.10) to the constant δ occurring in (2.3) resp. (2.7). For this we multiply (2.3) by $f\,q^2$ and integrate the resulting equation from 1 to q and then putting $q \to \infty$. By this procedure the term resulting from the square bracket of Eq. (2.3) vanishes and we have

$$\delta \int_1^\infty f\, q^2\, \mathrm{d}q = \int_1^\infty f\, \mathrm{d}q\,, \tag{2.15}$$

which by virtue of (2.11) can be written as follows:

$$\delta x_\infty = \int_0^{x_\infty} \frac{\mathrm{d}x}{q^2(x, A)}\,, \tag{2.16}$$

where x_∞ is defined as the value of $x = x(q, A)$ where q becomes infinite. Equation (2.16) then determines A for any given value of δ and hence for any Rayleigh number. This completes the solution.

3. The Solution of the Differential Equation for $y = y(x, A)$

What remains to be done is to solve the differential equation (2.10) numerically. The differential equation has two singular points, one at $x=0$, $y=1$, and the other at $x=\infty$ and $y=\infty$. All integral curves have to go through these two singular points. In the vicinity of the singular points, one can obtain analytic approximations for the solution of Eq. (2.10). The singular points have to be excluded from the numerical integration procedure, but a knowledge of the solution in the vicinity of the singular points is necessary. Since the solution of the differential Eq. (2.10) is physically meaningful only for $0 < x < x_\infty$, the behavior around the other singularity is not required if the integration is started from the singularity at $x=0$, $y=1$.

The integration of the differential equation was performed numerically by the Runge-Kutta method and the result of this calculation is plotted in Fig. 1

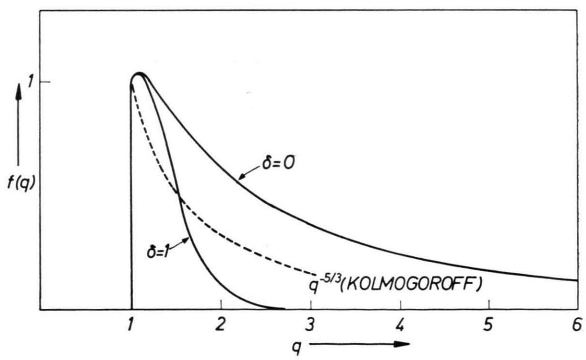

Fig. 1. The nondimensional energy spectrum for turbulent convection for $\delta=0$ ($R=\infty$) and $\delta=1$. The dotted line is the Kolmogoroff spectrum.

giving the spectral function $f(q)$ for $\delta=0$, that is $R=\infty$ and for $\delta=1$. Furthermore, in Fig. 2 the dependence of the constant of integration A on the number δ is plotted for the range of interest, given by $0<\delta<1$. It follows especially that for $\delta=0$, that is $R=\infty$ one has $A=5.2338$. The Kolmogoroff $q^{-5/3}$ power law is also plotted in Fig. 1 to see how the actual spectrum will deviate from it.

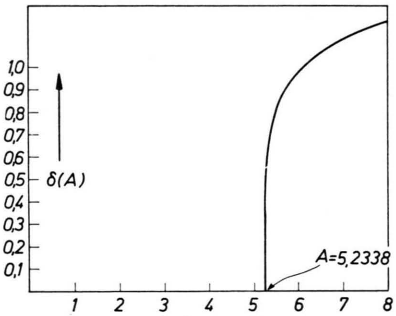

Fig. 2. The dependence of δ upon A.

Near the singularity at $x=0$, $y=1$ the Runge-Kutta solution is matched with the analytical approximation.

To obtain the analytic approximation near the singularity, we write the differential Eq. (2.10) as follows:

$$\mathrm{d}y/\mathrm{d}x = 4\,(y - \sqrt{y} - x)\,/x \qquad (3.1)$$

and expand the r.h.s. in the vicinity of the singular point located at $x=0$, $y=1$, with the result

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2\,y}{x} + \frac{2}{x} + 4 = 0\,. \qquad (3.2)$$

Eq. (3.2) is a first order linear differential equation with the general solution

$$y\,(x, A) = 1 + 4\,x - A\,x^2, \quad A>0\,. \qquad (3.3)$$

We have chosen the sign of the constant of integration in (3.3) as to make $y\,(x, A)$ zero for a certain value of x; otherwise, $q=q\,(x, A)$ as given by (2.13) will not cover the entire range $1>q>\infty$.

4. Some Expectation Values Derived from the Energy Spectrum

In the following we derive a number of expectation values of interest in applying our results to a realistic situation.

We are interested in the following mean or expectation values:

$$\langle v^2\rangle_k = \int_k^\infty F(k)\,\mathrm{d}k\,, \qquad (4.1)$$

$$\langle T'^2\rangle_k = \int_k^\infty G(k)\,\mathrm{d}k\,, \qquad (4.2)$$

$$\langle v\,T'\rangle_k = \int_k^\infty H(k)\,\mathrm{d}k\,. \qquad (4.3)$$

One can show that the expectation values (4.2) and (4.3) can be expressed by the expectation value (4.1). Eliminating H from (1.27) and (1.28) and observing that $\chi^* = \nu^*/\gamma$, we have

$$G = (\beta\,\gamma/g\,\alpha)\,F\,. \qquad (4.4)$$

From (1.29) and (4.4), it follows further

$$H = \sqrt{\beta\,\gamma/2\,g\,\alpha}\,F\,. \qquad (4.5)$$

hence

$$\langle T'^2\rangle_k = (\beta\,\gamma/g\,\alpha)\,\langle v^2\rangle_k\,, \qquad (4.6)$$

and

$$\langle v\,T'\rangle_k = \sqrt{\frac{\beta\,\gamma}{2\,g\,\alpha}}\,\langle v^2\rangle_k\,. \qquad (4.7)$$

With Eq. (1.38) and the definitions (2.1) and (2.2) we can write for (4.1), (4.6) and (4.7)

$$\langle v^2\rangle_q = g\,\alpha\,\beta\,\gamma\left(\frac{d}{2\,\pi\,\varkappa}\right)^2 \int_q^\infty f(q)\,\mathrm{d}q\,, \qquad (4.8)$$

$$\langle T'^2\rangle_q = \left(\frac{\beta\,\gamma\,d}{2\,\pi\,\varkappa}\right)^2 \int_q^\infty f(q)\,\mathrm{d}q\,, \qquad (4.9)$$

$$\langle v\,T'\rangle_q = \frac{(g\,\alpha)^{1/2}}{2}\left(\frac{d}{\varkappa\,\pi}\right)^2 \left(\frac{\beta\,\gamma}{2}\right)^{3/2} \int_q^\infty f(q)\,\mathrm{d}q\,. \qquad (4.10)$$

The function $\left[\int_q^\infty f(q)\,\mathrm{d}q\right]^{1/2}$ is plotted in Fig. 3.

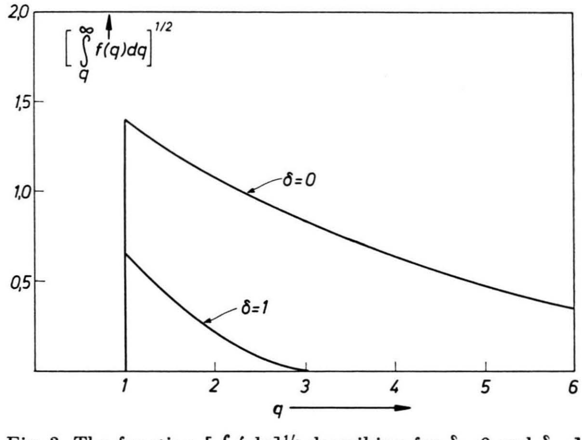

Fig. 3. The function $[\int f\,\mathrm{d}q]^{1/2}$ describing for $\delta=0$ and $\delta=1$ the fluctuations in velocity, temperature and the velocity-temperature correlation.

Of special interest are the total mean values obtained by putting in (4.8) – (4.10) $q=1$. The integral for $q=1$ has a certain numerical value which for $\delta=0$ is given by

$$\int_1^\infty f(q)\,\mathrm{d}q \cong 1.96\,. \qquad (4.11)$$

To apply these results, for example, to a convective cloud in the earth atmosphere we put for instance (c.g.s. units)

$$d = 10^5\,\text{cm}, \quad \alpha = 3\times 10^{-3}\,^\circ\text{K}^{-1},$$
$$\beta = 3\times 10^{-5}\,^\circ\text{K/cm}, \quad \gamma = 1.4, \tag{4.12}$$
$$\nu = 10^{-1}\,\text{cm}^2/\text{sec}, \quad g = 0.981\times 10^3\,\text{cm/sec}^2,$$
$$\varkappa = 0.33$$

and obtain

$$R = 1.37\times 10^{18} \tag{4.13}$$

furthermore

$$[\langle v^2\rangle_{q=1}]^{1/2} = 8\times 10^2\,\text{cm/sec},$$
$$[\langle T'^2\rangle_{q=1}]^{1/2} = 2.8\,^\circ\text{K}, \tag{4.14}$$
$$\frac{\langle v\,T'\rangle}{\chi\,\beta} = \frac{\gamma}{\nu\,\beta}\langle v\,T'\rangle = 3.7\times 10^8.$$

The value for the Rayleigh number (4.13), shows that we have to assume $\delta = 0$. The mean value of the velocity and temperature fluctuation (4.14) seems to be in good qualitative agreement with the observation. The third value of (4.14) indicates the predominance of the heat transfer by turbulent convection over that by conduction.

Finally, we are interested in the lifetime of an eddy belonging to the wave number k, respectively $q = k/k_0$. This lifetime is given by

$$\tau = 1/k[\langle v^2\rangle_q]^{1/2}. \tag{4.15}$$

With $k = k_0\,q$, $k_0 = \sqrt{2}\,\pi/d$ and the expression $\langle v^2\rangle_q$ Eq. (4.8) we obtain from (4.15)

$$\tau/\tau_0 = 1/q[\textstyle\int f\,dq]^{1/2},$$
$$\tau_0 = (2\,\varkappa^2/g\,\alpha\,\beta\,\gamma)^{1/2}. \tag{4.16}$$

The function τ/τ_0 is plotted in Fig. 4. For $q = 1$ one has $\tau/\tau_0 = 0.7$ and hence for the chosen values of

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the example given above

$$\tau = 29\,\text{sec} \tag{4.17}$$

as the characteristic lifetime.

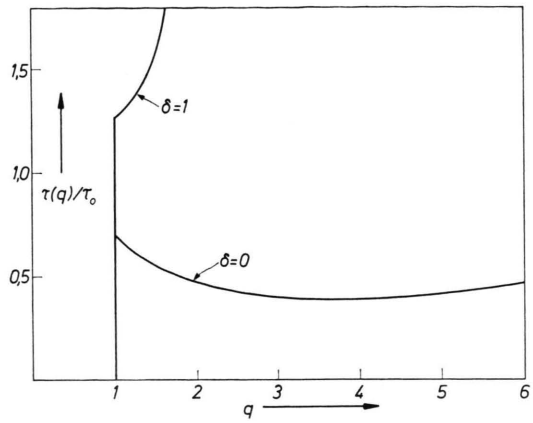

Fig. 4. The mean life time for an eddy of dimensionless wave number q.

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